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Asymptotic Solutions of the Continuous-Time Random Walk Model of Diffusion

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Asymptotic distributions of the Montroll–Weiss equation for the continuous-time random walk are investigated for long times. It is shown that, for a certain subclass of the hopping waiting time distributions belonging to the domain of attraction of stable distributions, these asymptotic distributions are of stable form. This indicates that the realm of applicability of the diffusion equation is limited. The Montroll–Weiss equation is rederived to include the influence of the initial waiting interval and the role of the stable distributions in physical problems is briefly discussed.

KEY WORDS: Random walk; diffusion; mobility; stable distributions.

1. INTRODUCTION

The Montroll–Weiss[(1)] equation (MW equation) is appropriate to a type of generalized diffusion which is directly applicable to a large class of diffusion problems.[(2)] For many problems it is equivalent to a master equation approach and for highly non-Markovian processes supersedes it.[(3,5)] It can be derived as follows.

Suppose a particle performs jumps such that the individual jump vector in space has a probability density $p(\mathbf{r})$ and that all jump vectors are statistically independent with identical distributions. The position relative to the

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origin of the process after exactly n jumps is deduced by noting that the characteristic function of the final distribution is equal to $\lambda^n(\mathbf{k})$, where

$$\lambda(\mathbf{k}) = \int p(\mathbf{r}) \exp(i\mathbf{k}\cdot\mathbf{r})\, d\mathbf{r} \tag{1}$$

and the integration is performed over all space. In general the time intervals between jumps will not be constant but will also be governed by a probability distribution $F(t)$. If all the intervals are independent and have identical distributions, it can be shown[6] that the probability of having N jumps in time t is given by

$$P\{N = n\} = F^{n*}(t) - F^{(n+1)*}(t) \tag{2}$$

where the symbol * indicates a convolution in the sense of Feller.[7] In Eq. (2) it is assumed that a jump has just taken place at $t = 0$; this is the situation relevant to a Green's function interpretation of the MW formula. On the other hand, for the situation where it is not known that a jump has taken place at $t = 0$, we must introduce the distribution of the first waiting interval, which will not usually be the same as $F(t)$. Denoting this by $H(t)$, we have

$$P\{N = n\} = H(t) * F^{(n-1)*}(t) - H(t) * F^{n*}(t) \tag{3}$$

If the particular time intervals are independent of their jump vectors, a simple randomization suffices to give the density of the final position at time t, given that at $t = 0$ the particle was located at $\mathbf{r} = 0$, i.e.,

$$p(\mathbf{k}, t) = \sum_{n=0}^{\infty} \lambda^n(\mathbf{k})[H * F^{(n-1)*} - H * F^{n*}]$$

On taking the Laplace transform with respect to time and performing a geometric sum, we obtain a generalized MW formula:

$$p(\mathbf{k}, s) = \frac{h(s)[1 - \psi(s)]}{s\psi(s)[1 - \lambda(\mathbf{k})\psi(s)]} \tag{4}$$

where

$$\psi(s) = \int_0^\infty e^{-st} F(dt) \qquad \text{and} \qquad h(s) = \int_0^\infty e^{-st} H(dt)$$

Clearly if $p(\mathbf{k}, s)$ is to represent a Green's function, then $\psi(s) = h(s)$ and some simplification results (we obtain the standard form).

It is important to note that in Eq. (4) position and time are not on the same footing: The double inversion will give the probability $p(\mathbf{r}, t)\, d\mathbf{r}$ of observing the particle at position $\mathbf{r}$ in volume $d\mathbf{r}$ at time t.

Solutions of the MW equation have been calculated numerically by Montroll and Sher,[8] while Shlesinger[9] has examined some asymptotic properties such as the mean position of the particle and the dispersion. The treatment relies on the use of Tauberian theorems which means that the distributions, particularly of $F(t)$, are regularly varying in the tail. $F(t)$'s belonging to the domain of attraction of a stable distribution with exponent ν are treated (the results presented here are entirely consistent with those of Shlesinger). For $0 < \nu < 1$ some success is gained in the explanation of observed mobilities in some amorphous materials. An example of the natural occurrence of distributions $F(t)$ of this type has been given by Tunaley.[10]

In the present study we examine the shape of the asymptotic distributions (or their tails) and show that these will be of stable form under suitable conditions of regular variation in the distributions $P(\mathbf{r})$, $F(t)$. The very important cases where $1 < \nu < 2$ are included. Although the stable densities cannot usually be expressed in closed form, they can be obtained numerically using the series expansion given by Feller,[7] for example.

2. SCALING

We must examine the shape of the distribution of the final position after long times such that a further increase in time produces a change in shape of such a type that the original shape can be recovered by a simple scale transformation. If the solution of the MW equation in one dimension is

$$P\{X \leq x; t\} = G(x, t) \tag{5}$$

we suppose that scale factors a and b exist so that the asymptotic solution $G'(x, t)$ is given by

$$\lim_{a,b \to \infty} P\{X/a \leq x; bt\} = G'(x, t) \tag{6}$$

Evidently from Eq. (5) we have

$$P\{X/a \leq x; bt\} = G(ax, bt)$$

and after differentiation the density of X/a becomes

$$p_{X/a,bt}(x, t) = ap_{x,t}(ax, bt)$$

The Fourier–Laplace transform of the new density is given by

$$\begin{aligned} p_{X/a,bt}(k, s) &= \int ap_{x,t}(ax, bt)e^{ikx-st}\,dx\,dt \\ &= (1/b)p_{x,t}(k/a, s/b) \end{aligned} \tag{7}$$

Because of the uniqueness properties of the Fourier–Laplace transform, Eqs. (6) and (7) imply that the transform of the asymptotic density $p'(k, s)$ is obtained from

$$p'(k, s) = \lim_{a,b \to \infty} (1/b)p(k/a, s/b)$$

The relationship between a and b has a definite form. Suppose $a = f(b)$ and consider two consecutive scalings. Clearly

$$a_1 a_2 = f(b_1 b_2) = f(b_1) f(b_2)$$

The solution to this is

$$f(b) = b^{\rho}, \qquad -\infty < \rho < \infty \tag{8}$$

The exponent ρ is chosen according to the particular distributions involved.

It is easily verified that the conclusions are similar in three-dimensional space and returning to the MW equation, it is seen that

$$p'(\mathbf{k}, s) = \lim_{a,b\to\infty} \frac{h(s/b)[1 - \psi(s/b)]}{s\psi(s/b)[1 - \lambda(\mathbf{k}/a)\psi(s/b)]} \tag{9}$$

As an example, we can treat the case of ordinary diffusion where the mean jump vector is zero. We will also assume that $\lambda(\mathbf{k})$ can be expanded in a series up to and including the second moment. For simplicity we also assume that the direction of the jump is uniformly distributed on a sphere so that it is symmetric and $\lambda(\mathbf{k})$ is only a function of k. Hence

$$\lambda(\mathbf{k}) = 1 - k^2\sigma_0^2/2 + \cdots \tag{10}$$

where σ_0^2 is the variance of the x component (say) of a single jump distance. Similarly $\psi(s)$ can be expanded

$$\psi(s) = 1 - \alpha s + \beta s^2/2 - \cdots \tag{11}$$

while, for the moment, we put $h(s) = \psi(s)$. In the limit $a, b \to \infty$ only the the first two terms are important and Eq. (9) yields

$$p'(\mathbf{k}, s) = \lim_{a,b\to\infty} \frac{\alpha}{\alpha s + (k^2\sigma_0^2 b/2a^2)} \tag{12}$$

By choosing the norming constant b equal to a^2 according to Eq. (8), we have

$$p'(\mathbf{k}, s) = (s + k^2\sigma^2{}_0/2\alpha)^{-1} \tag{13}$$

Inverting this with respect to s and k shows that

$$p(\mathbf{r}, t) = (4\pi Dt)^{-3/2} \exp(-r^2/4Dt) \tag{14}$$

where the coefficient $D = \sigma_0^2/2\alpha$. This result is consistent with the solution of the diffusion equation

$$\partial n/\partial t = D\, \nabla^2 n \tag{15}$$

for a δ-function at the origin at time $t = 0$ [this is most easily seen by taking the transform of Eq. (15)].

It is worthwhile noting that $p(\mathbf{r}, t)$ of Eq. (14) represents a normal density (stable with $\nu = 2$) in the position r, and interestingly also takes the form of a stable density with $\nu = \frac{1}{2}$ in t. We shall be more interested, however, in the marginal density representing diffusion along the x axis (say) when drifts, such as occur when a field is applied, are considered. The marginal density can be obtained from Eq. (13), for example, by simply putting $k_y = k_z = 0$, so that

$$p'(k_x, s) = (s + k_x^2\sigma_0^2/2\alpha)^{-1}$$

The inversion first with respect to k_x and then s gives

$$p'(x, t) = \frac{1}{2\pi i}\int \frac{\alpha \exp[-(2\alpha s)^{1/2}|x|/\sigma_0] \exp st\, ds}{\sigma_0(2\alpha s)^{1/2}}$$

Consider now $P\{X > x, t\}$. Evidently for $x \geq 0$

$$P\{X > x, t\} = \frac{1}{2\pi i}\int\int_x^\infty \frac{\alpha \exp[-(2\alpha s)^{1/2}x/\sigma_0] \exp st\, ds\, dx}{\sigma_0(2\alpha s)^{1/2}}$$

$$= \frac{1}{4\pi i}\int \frac{\exp[-(2\alpha s)^{1/2}x/\sigma_0] \exp st\, ds}{s}$$

The exponential factor in the integrand is the Laplace transform of a stable density with exponent $\nu = \frac{1}{2}$. To remove the scale factor $(2\alpha x^2/\sigma_0^2)^{1/2}$, transformations can be performed $X \to X/\sigma_0$, $t \to t/2\alpha x^2$ in order to express the result in terms of a stable distribution (the s^{-1} term converts the density into a distribution) with unit scale factor S_ν. Thus for $x > 0$

$$P\{X/\sigma_0 > x, t\} = \tfrac{1}{2}S_{1/2}(t/2\alpha x^2) \tag{16}$$

A similar development for $x < 0$ shows that

$$P\{X/\sigma_0 < x, t\} = \tfrac{1}{2}S_{1/2}(t/2\alpha x^2) \tag{17}$$

which indicates the required symmetry. It will be shown that the expression obtained above, though more cumbersome than the normal density, appears more fundamental in generalized diffusion.

3. A GENERALIZED SYMMETRIC DIFFUSION

In this section we assume that the jump vectors belong to the domain of attraction of the normal distribution but that the time intervals belong to the domain of attraction of stable distributions. First we take $0 < \nu < 1$ and limit $F(t)$ to be of the asymptotic form

$$1 - F(t) \sim \alpha/t^\nu\Gamma(1 - \nu), \qquad t \to \infty \tag{18}$$

where α is a constant. This class of $F(t)$ is not as general as that treated by Feller in that α is not a slowly varying function of t (e.g., $\alpha = \ln t$). The choice $0 < \nu < 1$ means that the jumping time intervals have infinite mean and variance. It can be shown[7] that Eq. (18) implies

$$1 - \psi(s) \sim \alpha s^{\nu}, \qquad s \to 0 \tag{19}$$

and this can be employed to give the expansion of $\psi(s)$ required for deriving the asymptotic form of $P\{X > x, t\}$, etc.

Inserting the expansions into Eq. (9) yields for the Green's function

$$p'(\mathbf{k}, s) = \lim_{a, b \to \infty} \frac{\alpha s^{\nu - 1}}{b^{\nu}[(\alpha s^{\nu}/b^{\nu}) + (k^2 \sigma_0{}^2/2a^2)]} \tag{20}$$

Choosing $a^2 = b^{\nu}$, we obtain

$$p'(\mathbf{k}, s) = s^{\nu - 1}/(s^{\nu} + k^2 \sigma_0{}^2/2\alpha) \tag{21}$$

The marginal density $p'(x, t)$ can be derived by inversion and proceeding as above for $x > 0$:

$$P\{X > x, t\} = \frac{1}{4\pi i} \int \frac{\exp\{[-(2\alpha s^{\nu})^{1/2}/\sigma_0]x\}}{s} \exp st \, ds \tag{22}$$

Thus for $x > 0$

$$P\left\{\frac{X}{\sigma_0} > x, t\right\} = \frac{1}{2} S_{\nu/2}\left(\frac{t}{(2\alpha x^2)^{1/\nu}}\right) \tag{23}$$

while for $x < 0$

$$P\left\{\frac{X}{\sigma_0} < x, t\right\} = \frac{1}{2} S_{\nu/2}\left(\frac{t}{(2\alpha x^2)^{1/\nu}}\right) \tag{24}$$

The presence of the longer tail in the case of $S_{\nu/2}$ as opposed to $S_{1/2}$ indicates that the particle is more likely to be found near the origin. Comparing the transform in Eq. (21) with that of Eq. (15), it is clear that the diffusion equation is inapplicable to the present situation.

The next case of interest concerns distributions $F(t)$ which yield a finite mean but infinite variance in the jumping time interval. This is concentrated on the positive half axis (negative times are not allowed) and an infinite variance implies a long tail for large times. The density must be asymmetric and be rapidly decreasing toward the origin. Although this type of distribution is one-sided, it will be convenient to transform out the centering and deal with a two-sided density. Thus a Fourier rather than Laplace transform is required in time and defining

$$\psi(\omega) = \int e^{i\omega t} F(dt) \tag{25}$$

yields a MW equation:

$$p(\mathbf{k}, \omega) = \frac{h(\omega)[1 - \psi(\omega)]}{-i\omega\psi(\omega)[1 - \lambda(\mathbf{k})\psi(\omega)]} \tag{26}$$

The scaling theory and the derivation of the asymptotic forms remain similar.

For the type of distribution under consideration, the characteristic function expansion is chosen to be of the form[7] ($\omega \to 0$)

$$\psi(\omega) \sim 1 + i\omega\alpha + \beta|\omega|^{\nu} e^{\pm i\pi\nu/2} + \cdots, \qquad 1 < \nu < 2 \tag{27}$$

where the positive sign applies when $\omega < 0$ and the negative sign when $\omega > 0$. The constant α represents the mean time between jumps and β is a scale parameter. Clearly $F(t)$ belongs to the domain of attraction of a stable distribution with exponent ν, though due to the centering term $i\omega\alpha$ it is not strictly stable. Using the expansion of $\lambda(k)$ for one-dimensional diffusion in Eq. (10), the transform of the asymptotic form of the Green's function is given by

$$p'(k, \omega) = \lim_{a,b\to\infty} \frac{\alpha}{b[(-i\omega\alpha/b) + (k^2\sigma_0{}^2/2a^2)]} \tag{28}$$

The term in $|\omega|^{\nu}/b^{\nu}$ becomes negligible compared with that in ω/b since $\nu > 1$. Choosing $a = b^{1/2}$, we have

$$p'(k, \omega) = [(k^2\sigma_0{}^2/2\alpha) - i\omega]^{-1} \tag{29}$$

which on performing the double inversion gives a normal density in x or the same result as Eq. (16) and (17) for ordinary diffusion.

4. ASYMMETRIC DIFFUSION

This applies typically to a situation when the diffusing particles are subjected to a constant field so that the random walk is biased in a direction of increasing x (say). Thus the probability of jumping forward is greater than that of jumping backward and this means that $\lambda(k)$ must include a term representing a finite mean distance μ traveled at each jump. The expansion of $\lambda(k)$ is now

$$\lambda(k) \sim 1 + ik\mu - k^2\sigma^2/2 \tag{30}$$

Using the expansion for $\psi(s)$ given by Eq. (11), which is appropriate to ordinary diffusion, and inserting these into Eq. (9) gives the trivial result

$$p'(k, s) = \lim_{a,b\to\infty} \frac{\alpha}{b[(\alpha s/b) + (ik\mu/a)]} \tag{31}$$

$$= (s + ik\mu/\alpha)^{-1} \tag{32}$$

where we have chosen $a = b$. The double inversion yields

$$p'(x, t) = \delta(x - \mu t/\alpha) \tag{33}$$

which is an improper probability density which simply represents the drift part of the behavior: The drift velocity is μ/α. To obtain a meaningful density, we can transform away the drift motion and introduce a drifting coordinate system so that the density is always centered at the origin. Thus we examine $P\{Y > x, t\} = P\{X - \mu t/\alpha > x, t\}$ and proceed to determine the effect of this transformation on the MW equation. Evidently

$$P\{Y > x, t\} = G(x + \mu t/\alpha, t)$$

with density

$$p_y(x, t) = p_x(x + \mu t/\alpha, t) \tag{34}$$

The Fourier–Laplace transform of $p_y(x, t)$ is

$$\begin{aligned} p_y(k, s) &= \int p_x(x + \mu t/\alpha, t) e^{ikx - st}\, dx\, dt \\ &= p_x(k, s + ik\mu/\alpha) \end{aligned} \tag{35}$$

In the asymptotic form of $p_y(k, s)$ an examination of the behavior of the denominator of Eq. (9) shows that we must include the first three terms of the expansions of $\psi(s)$ and $\lambda(k)$ [Eqs. (11) and (30)]. The denominator, $1 - \lambda(k/a)\psi(s/b)$, becomes

$$-\frac{ik\mu}{a} + \frac{k^2\sigma^2}{2a^2} + \frac{\alpha s}{b} + \frac{ik\mu\alpha s}{ab} - \frac{\alpha s k^2\sigma^2}{2a^2 b} - \frac{\beta s^2}{2b^2} - \frac{ik\mu\beta s^2}{2ab^2} + \frac{\beta s^2 k^2 \sigma^2}{4a^2 b^2}$$

Letting $s \to s + ik\mu/\alpha$ and keeping only those terms with the lowest exponents of a and b yields

$$\frac{\alpha s}{b} + \frac{k^2}{2a^2}\left(\sigma^2 - 2\mu^2 + \frac{\mu^2\beta}{\alpha^2}\right) \tag{36}$$

Thus

$$p_y'(k, s) = \lim_{a, b \to \infty} \left\{\alpha/b\left[\frac{\alpha s}{b} + \frac{k^2}{2a^2}\left(\sigma^2 - 2\mu^2 + \frac{\mu^2\beta}{\alpha^2}\right)\right]\right\} \tag{37}$$

Letting $a = b^{1/2}$ yields

$$p_y'(k, s) = \left[s + \frac{k^2}{2\alpha}\left(\sigma^2 - 2\mu^2 + \frac{\mu^2\beta}{\alpha^2}\right)\right]^{-1} \tag{38}$$

Again this represents a normal distribution in x but the effective diffusion coefficient is now

$$D = (1/2\alpha)(\sigma^2 - 2\mu^2 + \mu^2\beta/\alpha^2) \tag{39}$$

To compare this with symmetric diffusion we can note that when $\mu = 0$ the ordinary diffusion coefficient is obtained. The constant β is the mean square jump interval and $\beta \geq \alpha^2$; with no dispersion in the jumping interval $\beta = \alpha^2$ and

$$D = (1/2\alpha)(\sigma^2 - \mu^2) \tag{40}$$

Since $\sigma^2 - \mu^2$ is the variance in the jump position vector, in effect the simple diffusion coefficient is obtained. Clearly an uncertainty in the jumping interval increases the variance of the drifting walk.

The treatment can be extended to the more general cases and for $0 < \nu < 1$ the expansion in Eq. (19) can be employed

$$p_x'(k, s) = \lim_{a,b \to \infty} \left[\alpha s^{\nu-1}/b^\nu \left(\frac{\alpha s^\nu}{b^\nu} - \frac{ik\mu}{a} + \frac{k^2\sigma^2}{2a^2} \right) \right] \tag{41}$$

To obtain the asymptotic form, we must let $a = b^\nu$ and the term $k^2\sigma^2/2a^2$ becomes negligible. Hence

$$p_x'(k, s) = \frac{s^{\nu-1}}{s^\nu - ik\mu/\alpha} \tag{42}$$

The inversion with respect to k can be performed by noting that with s on the imaginary axis the pole lies in the lower half of the complex plane:

$$p_x(x, s) = (\alpha s^{\nu-1}/\mu) \exp(-\alpha x s^\nu/\mu)$$

This results in

$$P\left\{\frac{X}{\mu} > x, t\right\} = S_\nu\left(\frac{t}{(\alpha x)^{1/\nu}}\right), \qquad x > 0 \tag{43}$$

while for $x < 0$ there is no contribution.

It is interesting to note that no change of axes is required and that the asymptotic behavior depends critically on whether or not a drift is present [compare Eqs. (23) and (24)].

For $1 < \nu < 2$ the Fourier approach is required using the expansion in Eq. (27). A brief examination will show that, in the first approximation,

$$p_x'(x, t) = \delta(x - \mu t/\alpha) \tag{44}$$

so that the Fourier analog of Eq. (35) must be employed, i.e.,

$$p_y(k, \omega) = p_x(k, \omega - k\mu/\alpha)$$

On performing the above transformation and dropping high-order terms, the term in the denominator of the asymptotic expansion becomes

$$-\frac{i\alpha\omega}{b} - \beta e^{\pm i\pi\nu/2} \left| \frac{\omega}{b} - \frac{k\mu}{a\alpha} \right|^\nu$$

Thus

$$p_y'(k, \omega) = \lim_{a,b \to \infty} \left\{ \alpha/b \left[-i\alpha \frac{\omega}{b} - \beta e^{\pm i\pi\nu/2} \left| \frac{\omega}{b} - \frac{k\mu}{a\alpha} \right|^{\nu} \right] \right\} \tag{45}$$

We choose $a = b^{1/\nu}$ and recall $\nu > 1$; as $a, b \to \infty$ the term $k\mu/a\alpha$ will eventually dominate the ω/b term in the modular brackets. Therefore

$$p_y'(k, \omega) = i\left(\omega - i\frac{\beta}{a} e^{\pm i\pi\nu/2} \left| \frac{k\mu}{\alpha} \right|^{\nu}\right)^{-1} \tag{46}$$

where the positive sign is taken when $k > 0$ and the negative sign when $k < 0$. Whether k is positive or negative, the pole in ω lies in the lower half of the complex plane, so that the inversion with respect to ω can be performed easily:

$$p_y(k, t) = \exp\left[\frac{\beta t}{\alpha} e^{\pm i\pi\nu/2} \left| \frac{k\mu}{\alpha} \right|^{\nu}\right] \tag{47}$$

It can be shown[7] that this is the characteristic function of an asymmetric stable distribution with exponent ν with a long tail on the negative half-axis and short tail on the positive half-axis. However, it is worth pointing out that while the mean position varies linearly with time, the dispersion only increases at a rate proportional to $t^{1/\nu}$ by virtue of the properties of stable densities.

The result can be expressed in terms of the stable distribution to which domain the distribution $F(t)$ belongs: Let this be $S_\nu(t; \alpha, \beta)$, where α and β are the same constants appearing in Eq. (27). The mean value, corresponding to α, is found from Eq. (44), namely $\mu t/\alpha$. The second constant which may be determined by expanding the rhs of Eq. (47) is equal to $\beta t\mu^\nu/\alpha^{1+\nu}$. However, we must take into account that the long tail in the distribution of Y is in the opposite direction to that in t, so that

$$P\{Y > -x\} = S_\nu(x; 0, \beta t\mu^\nu/\alpha^{1+\nu})$$

In terms of the stable distribution with unit scale parameter and zero centering we have

$$P\left\{Y > -x\left(\frac{\beta t}{\alpha}\right)^{1/\nu} \frac{\mu}{\alpha}\right\} = S_\nu(x)$$

$$P\left\{X > \frac{\mu t}{\alpha} - \frac{\mu x}{\alpha}\left(\frac{\beta t}{\alpha}\right)^{1/\nu}\right\} = S_\nu(x)$$

This result is entirely equivalent to that derived by Feller[7] for a simple situation. Once again the asymptotic distributions take on entirely different forms depending on whether there is a drift: Without drift the distribution is normal in x even when $1 < \nu < 2$.

5. DISCUSSION

So far the possible influence of the initial waiting time distribution has not been taken into account. A situation where this should be considered is where particles are dumped into a material at time $\tau = 0$ and are allowed to diffuse. Measurements are started at some time τ (where $t = 0$) so that it would not be certain that a particle had just jumped into its observed location. We have seen that this introduces a factor $h(s)/\psi(s)$ into the MW equation and in the limit $b \to \infty$ this reduces to $h(0)$. If $H(t)$ is not defective, $h(0) = 1$ and the results obtained above are equally applicable to the generalized case. When the mean time between jumps is finite Feller[7] shows that $H(t)$ is a proper distribution, so that $h(0) = 1$: thus when $1 < \nu \leq 2$ there is no effect. When $0 < \nu < 1$, Feller shows that a generalized arc sine distribution is appropriate. This has a very long tail but it can readily be verified that it is nevertheless a proper distribution and again $h(0) = 1$. On the other hand, we may have to wait a time $t \gg \tau$ before the asymptotic distributions derived above are a reasonable approximation; of course this is tantamount to ignoring the initial waiting time distribution; but to achieve a better result, the Laplace transform $h(s)$ is required and it appears that it may be difficult to express this in closed form.

We have derived the asymptotic forms appropriate to a wide range of distributions for the waiting interval with and without drift. The jump vector is supposed to have a finite variance and applications where this latter condition must be relaxed are probably few. Although the most interesting case is when $0 < \nu < 1$, the area of greatest application (for example, in diffusion and mobility considerations concerning amorphous materials) is probably $1 < \nu < 2$. This is because the mean time between jumps is finite so that macroscopic quantities such as mobility and electrical resistance are also finite and their mean values will not be a function of time, although wild variations are possible.

Finally we note that the diffusion equation has been remarkably successful in the interpretation of certain physical phenomena but its solution is really an asymptotic form of the MW equation. This is because many jumps occur on practical time scales. Similarly one might expect the generalized versions to be equally significant but it is unlikely that convergence to the derived distributions will occur at the same rate as for ordinary diffusion, so that some care must be exercised.

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